

Link Delay Estimation via Expander Graphs

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Abstract

In network tomography, we seek to infer the status of parameters (such as delay) for links inside a network through *end-to-end* probing between (external) boundary nodes along chosen routes. In this work, we apply concepts from compressed sensing to establish conditions on the routing matrix under which it is possible to estimate link delay from end-to-end measurements and also provide an upper-bound on the estimation error. Further, we investigate choice of the appropriate set of paths within a given network with defined boundary nodes, so as to minimize the number of probes needed for estimation. Simulation results show that the proposed algorithm will, with high probability, achieve forty percent reduction in overhead required compared to probing between every available pair of boundary nodes.

Index Terms

Network Tomography, Compressed Sensing, Graph Theory, Expander Graphs, Optimization

I. INTRODUCTION

Monitoring of link properties (delay, loss rates etc.) within the Internet continues to be an integral requirement within any network management framework, as part of monitoring its utilization and performance. Within the context of an ever-expanding Internet, the need for accurate and fast monitoring schemes has escalated in recent years due to the increasing popularity of new resource-consuming services (such as video-conferencing, Internet telephony, and on-line games) that require quality-of-service (QoS) guarantees. In 1996, the term *network tomography* was coined by Vardi [1] to encompass a class of approaches that seek to infer internal link status from end-to-end measurements. A useful classification of network tomography methods for our purposes is as follows [2]:

- **Cooperative Internal Nodes:** assumes presence of internal nodes on a route that respond to *control* packets. For example, active probing tools such as ping or traceroute, measure and report attributes of the round-trip path from external sender to internal node based on separate probe packets [3]. The challenges of such methods arise from the fact that many service providers do not own the entire network and hence do not have access to the desired internal nodes [4], [5].
- **End-to-End:** In networks with a defined *boundary*, it is assumed that access is available to all nodes at the edge (and not to any in the interior). A boundary node sends probes to all (or a subset) of other boundary

nodes to measure packet attributes on the path between network end points. Clearly, these edge-based methods do not require exchanging control messages with any interior nodes. The primary challenge confronting such end-to-end probe based link status estimation is that of identifiability, as will be discussed later [6], [7].

As the Internet evolves towards more decentralized, uncooperative and heterogeneous administrative (sub)domains, availability of cooperative interior nodes will be increasingly limited. Hence end-to-end network diagnostic tools are attracting increasing attention as a result. In end-to-end network tomography, probes are sent between two boundary nodes on *pre-determined* routes; typically these are the shortest paths between the nodes based on existing internet routing protocols. For parameters like delay, an additive linear model adequately represents the relation between measured path and individual link delay, and can be written as [8], [9]

$$\mathbf{y} = \mathbf{R}\mathbf{x} \quad (1)$$

where \mathbf{x} is the $n \times 1$ (unknown) vector of individual link delay. The $r \times n$ *binary* matrix \mathbf{R} is the routing matrix for the network graph corresponding to the paths chosen for the probes (each row of the matrix is a path) and $\mathbf{y} \in \mathbb{R}^r$ is the measured r -vector of end-to-end path delays. Although, the focus of this paper is link delay, our approach readily applies to any other link attributes (such as packet lost rate) which allows such a linear relation with end-to-end measurements.

Link delay estimators based on Eq. (1) can be classified under:

- 1) **Deterministic**: the delays are considered unknown but constant. Since the link delay is typically time varying, such approaches are suitable for periods of local ‘stationarity’ where such an assumption is valid.
- 2) **Stochastic**: the link delay vector \mathbf{x} is specified by a suitable a-priori parametric probability distribution; the method then estimates the unknown parameters of the model. For example, [10], [8], [9], [11] assume that link delay follow a Gaussian or an exponential distribution.

There exist challenges with both modeling approaches. Stochastic models are usually more computationally intensive than deterministic ones [12] as they suffer from over-modeling (too many parameters for the data). Moreover, in many scenarios, one is typically interested in only the *few* links that are congested (i.e. excessive link delay). Deterministic models are better suited to exploit this (side) information; our method falls within this class.

In Eq. (1), typically, the number of observations $r \ll n$, because the number of accessible boundary nodes is much smaller than number of links inside the network. Thus the number of variables in Eq. (1) to be estimated is much larger than the number of equations [11], leading to generic non-uniqueness of solutions to Eq. (1), i.e., inability to uniquely determine link delay [10] from end-to-end measurements.

A network administrator is typically interested in identifying only the (few) links with large delays (or high packet lost rate) at any given time; this side information suggests mechanisms to solve the under determined system in Eq. (1) provided that the *sparsity* of the desired solution can be exploited by the method. In other words, we are interested in solution vectors \mathbf{x} with only a few large entries (say, up to k). We refer to such vectors as *k-sparse*. We will show that by using the concept of *expander graphs* and compressed sensing, *k-sparse* delay vectors may be successfully estimated, provided some conditions on the routing matrix of a network are met. The estimates

obtained satisfy a desirable property, i.e. the difference between the true delay and the estimate (solution from Eq. (1)) goes to zero. We call such networks k -identifiable and show that Eq. (1) can be solved using a linear programming (LP) techniques.

Clearly, the choice of paths in the network used for probing directly affects the estimation accuracy but also constitutes an overhead. Hence, for a given estimation accuracy, one could attempt to minimize total number of probes used for network monitoring. We study this problem and arrive at a binary integer programming formulation. Given that this is an NP-hard problem, a heuristic algorithm is developed that provides good approximate results with reasonable complexity.

In summary, our specific contributions are as follows:

- We establish a novel connection between network tomography and binary compressed sensing via the notion of expander graphs;
- We provide conditions on the routing matrix of a network under which it is ($k =$) 1-identifiable. Moreover, we provide an upper-bound on estimation error in such cases.
- Based on our result for 1-identifiability, we show how to choose the routing matrix for a given network such that it is 1-identifiable, with minimum number of end-to-end transmitted probes.
- We provide simulation evidence for delay estimation when the delay vector \mathbf{x} in Eq. (1) is k -sparse for $k \geq 1$.

A preliminary version of our results appears in [13]. This submission is distinct due to the complete proofs of theorems, fuller discussion of network coding design trade-offs and a thorough set of simulation experiments. In addition in this manuscript, we propose an optimization problem which gives the minimum number of paths making the network 1-identifiable. That would decrease the cost of end-to-end measurement in a network.

As is customary, a network consisting of bidirectional links connecting transmitters, switches, and receivers can be modeled as an undirected graph $N(V, E)$ where V (E) is the set of vertices (edges). A set of measurements \mathbf{y} are obtained by end-to-end probing, given by Eq. (1). We provide analytical results for identifiability for the case of $k = 1$, which is sufficiently challenging (and unsolved to date). For case $k > 1$, we provide simulation results which show that for Internet-topology networks, the delay estimation method for $k = 1$ also typically yields acceptable estimation error.

The paper is organized as follows: Section II relates the routing matrix of a network to bipartite graphs. Section III establishes a connection between link delay estimation and binary compressed sensing and identifies conditions on network routing matrix under which a given network is 1-identifiable. In addition, we describe an algorithm for determining the expansion factor and error parameter of an expander graph. In Section IV, we look at minimizing number of probes sent subject to 1-identifiability of the network. We evaluate our findings using simulations in Section V. The paper concludes with reflections on future work in Section VI. Proof of all theorems are given in Appendix-A.

Notations: We use bold capitals (e.g. \mathbf{R}) to represent matrices and bold lowercase symbols (e.g. \mathbf{x}) for vectors. The i -th entry of a vector \mathbf{x} is denoted by x_i . For the matrix \mathbf{R} , $\mathcal{N}(\mathbf{R})$ denotes its Null space, and superscript t denotes its transpose. A set is denoted by a calligraphic capitalized symbol, e.g. \mathcal{R} and $|\mathcal{R}|$ is the cardinality

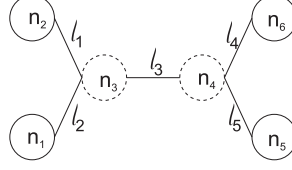


Fig. 1. A network with 4 boundary nodes, 2 intermediate nodes and 5 links

(number of elements) in the set. For any set $S \subset \{1, 2, 3, \dots, n\}$, S^c represents the complement. Also, for any vector $\mathbf{x} \in \mathbb{R}^n$, vector $\mathbf{x}_S \in \mathbb{R}^n$ has entries defined as follows:

$$(x_S)_i = \begin{cases} x_i & \text{if } i \in S \\ 0 & \text{o.w.} \end{cases} \quad (2)$$

If $\mathbf{x} \in \mathbb{R}^n$, the l_p -norm of \mathbf{x} is defined as follows:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \quad (3)$$

II. ROUTING MATRIX AND BIPARTITE GRAPH

In this section we show that the routing matrix of any network can be represented as a *bi-adjacency matrix* of a suitably defined *bipartite* graph. This will help connect the problem of network identifiability with *expander graphs*, a special subset of bipartite graphs. Throughout this manuscript, boundary nodes are depicted as solid circles while intermediate nodes are presented using dashed circles. The network in Figure 1 is used to illustrate the subsequent definitions.

Definition 1. A bipartite graph is one whose vertices can be divided into two disjoint sets X and Y such that every edge connects a vertex in X to one in Y [14].

A bipartite graph is usually represented as a triple $G(X, Y, H)$ where $H \subset X \times Y$ is a set with paired elements from X, Y . The vertex sets X and Y are called left and right sides of the graph, respectively. A bipartite graph $G(X, Y, H)$ can be represented by its *bi-adjacency matrix* $A = [a_{ij}]$, where $a_{ij} = 1$ if node $i \in X$ is connected to node $j \in Y$, and is zero otherwise, i.e.,

$$a_{ij} = \begin{cases} 1 & (i, j) \in H \\ 0 & (i, j) \notin H \end{cases} \quad (4)$$

$$\mathbf{A} = [a_{ij}]$$

Assume that a given network $N(V, E)$ has a total of n links (i.e., $n = |E|$), \mathcal{R} is the (chosen) set of paths between the boundary nodes of the network and $r = |\mathcal{R}|$. Let $\mathbf{R}_{r \times n}$ denote the routing matrix, where there exists an isomorphism between the set \mathcal{R} and the corresponding routing matrix \mathbf{R} . For example, for the network in Figure 1, suppose the following routing matrix is given:

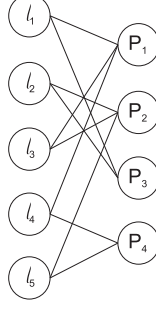


Fig. 2. Bipartite graph corresponding to given routing matrix in Eq. (5)

$$\mathbf{R} = \begin{array}{l} P_1 : n_2 \rightsquigarrow n_6 \\ P_2 : n_1 \rightsquigarrow n_5 \\ P_3 : n_1 \rightsquigarrow n_2 \\ P_4 : n_5 \rightsquigarrow n_6 \end{array} \begin{array}{c} l_1 \quad l_2 \quad l_3 \quad l_4 \quad l_5 \\ \left[\begin{array}{ccccc} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \end{array} \quad (5)$$

which is equivalent to the following set of paths \mathcal{R} :

$$\mathcal{R} = \{l_1 l_3 l_4, l_2 l_3 l_5, l_1 l_2, l_4 l_5\} \quad (6)$$

$\mathbf{R}_{r \times n}$ can be thought of a bi-adjacency matrix of a bipartite graph $G(X, Y, H)$ where $X = E$ (set of links in the network) and $Y = \mathcal{R}$ (set of given paths in the network). There exists a connection between a node in X to a node in Y if a path in Y includes the corresponding link in X . Figure 2 presents the bipartite graph for the network in Figure 1 with routing matrix \mathbf{R} in Eq. (5).

Note that the above routing matrix, or its equivalent set of paths, is not a complete set of routes for the network in Figure 1 (e.g., doesn't include the path from n_1 to n_6 which is $l_2 l_3 l_4$). However, it is a fundamental premise in network tomography that the routing matrix is already chosen, and may not be changed. Hence initially we seek to investigate the question: assuming that the routing matrix is given, when is it possible to identify or estimate a single ($k = 1$) link parameter? In Section IV, we re-visit this question under the assumption that the routing matrix may be freely chosen.

III. EXPANDER GRAPHS AND NETWORK IDENTIFIABILITY

In recent years, a new approach - *Compressed Sensing* - for estimating an n -dimensional (signal) vector \mathbf{x} from a lower dimensional representation - has attracted much attention [15], [16], [17]. For any signal $\mathbf{x} \in \mathbb{R}^n$, the reduced dimension representation is equal to $\mathbf{y} = \mathbf{A} \mathbf{x}$, where $m \times n$ matrix \mathbf{A} ($m \ll n$) is referred to as the *measurement matrix*. The main challenge in traditional compressed sensing is to construct \mathbf{A} with the following desirable (and conflicting) properties: a) achieve maximum possible compression (m/n small) and yet allow b) accurate reconstruction of \mathbf{x} from \mathbf{y} when \mathbf{x} is known to be sparse, using c) a fast decoding algorithm [18], [19],

[20], [21]. For example, when \mathbf{A} is a binary matrix, it is known that $m = O(k \log \frac{n}{k})$ suffices when \mathbf{x} is k sparse. In scenarios where the routing matrix is already pre-determined, the main issue is to determine whether it is an appropriate measurement matrix for compressive sensing, i.e., if it satisfies objective b) above. To the best of the authors' knowledge, there is no existing literature regarding this problem.

We first present a result on compressed sensing using expander graphs. Then, we establish a connection between $k = 1$ identifiability in a network $N(V, E)$ to the 'expansion factor' of an expander graph. This makes it possible to use results from compressed sensing to address the link identifiability problem.

A. Expander Graphs

Definition 2. A bipartite graph $G(X, Y, H)$ with left degree d (i.e. $\deg(v) = d \forall v \in X$) is a (ϕ, d, ϵ) -expander, if for any $\Phi \subset X$ with $|\Phi| \leq \phi$ the following condition holds:

$$|N(\Phi)| \geq (1 - \epsilon)d |\Phi| \quad (7)$$

where $N(\Phi)$ is set of neighbors of Φ . Parameters ϕ and ϵ are referred to as expansion factor and error parameter, respectively.

Roughly speaking, in an expander graph, the degree of connectivity for any collection of nodes in the left hand side (X) expands to a sufficiently large number of nodes in the right hand side (Y) [22]. Expander graphs are well-studied; [23], [24], [25] show how to construct a (ϕ, d, ϵ) -expander graph. In a key result, Berinde and Indyk in [26], [27] show that the bi-adjacency matrix of a $(2\phi, d, \epsilon)$ -expander graph can be used as the measurement matrix for a ϕ -sparse signal, for $\epsilon < \frac{1}{16}$. The parameter ϵ in an expander graph is a design variable which is related to recovery error. For our application, we use a larger value of ϵ as it expands the space of expander graphs and hence increases the likelihood that it leads to an identifiable network. On the other hand, increasing ϵ also increases the bound on error recovery and reduces the fidelity of the reconstruction. There is a consequent tradeoff between number of identifiable networks and the estimation error in link delay. In this paper, we focus on 1-identifiable networks or equivalently 1-sparse delay vectors. We show that for such networks, $\epsilon = 1/4$ yields acceptable tradeoff between number of 1-identifiable networks and estimation error. As will be shown in Section V, the principal ideas behind our results for $k = 1$ appear to generalize to $k > 1$ -sparse vectors.

The next lemma characterizes the null space of bi-adjacency matrix of an expander graph and will be used to bound the error in the recovery of \mathbf{x} from its compressive projection \mathbf{y} .

lemma 1. Let $G(V_1, V_2, E)$ be a $(2, d, \epsilon)$ -expander with $\epsilon \leq 1/4$ and $\mathbf{A}_{m \times n}$ be its bi-adjacency matrix. Assume \mathbf{w} lies in the null space of \mathbf{A} (i.e. $\mathbf{A}\mathbf{w} = \mathbf{0}$) and let S be any singleton set of coordinates of \mathbf{w} ; i.e. $S = \{i\}$, $i \in \{1, \dots, n\}$. Then

$$\|\mathbf{w}_S\|_1 \leq 2\epsilon \|\mathbf{w}_{S^c}\|_1 \quad (8)$$

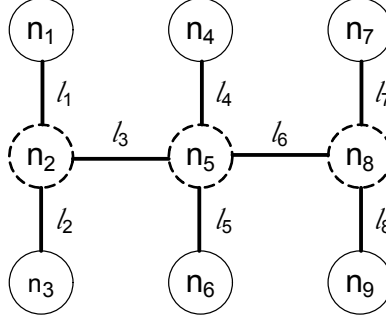


Fig. 3. A toy network with 6 boundary nodes, 3 interior nodes and 8 links. Its routing matrix, given in Eq. (11), satisfies expansion property.

Proof: See Appendix A.

The following theorem provides an upper bound on error of recovering \mathbf{x} from its lower dimensional projection \mathbf{Ax} when \mathbf{A} is a bi-adjacency matrix of an $(2, d, \epsilon)$ -expander graph and $\epsilon \leq 1/4$.

Theorem 1. Consider any two vectors \mathbf{x}, \mathbf{x}' , such that they have the same projection under measurement matrix \mathbf{A} ; i.e. $\mathbf{Ax} = \mathbf{Ax}'$. Further, without loss of generality, suppose $\|\mathbf{x}'\|_1 \leq \|\mathbf{x}\|_1$. Let S be the set of largest (in magnitude) elements of \mathbf{x} . Then

$$\|\mathbf{x}' - \mathbf{x}\|_1 \leq f(\epsilon) \|\mathbf{x}_{S^c}\|_1, \quad (9)$$

where

$$f(\epsilon) = \frac{2(1+2\epsilon)}{1-2\epsilon}, \quad \epsilon \leq \frac{1}{4}. \quad (10)$$

Proof: See Appendix A.

Example:

In this example, we show how the above result may be applied to a network like the one in Figure 3. Suppose we are given the following routing matrix:

$$\mathbf{R} = \begin{matrix} & \begin{matrix} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 \end{matrix} \\ \begin{matrix} P_1 : n_1 \rightsquigarrow n_3 \\ P_2 : n_1 \rightsquigarrow n_4 \\ P_3 : n_3 \rightsquigarrow n_6 \\ P_4 : n_4 \rightsquigarrow n_7 \\ P_5 : n_6 \rightsquigarrow n_9 \\ P_6 : n_7 \rightsquigarrow n_9 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{matrix} \quad (11)$$

We show that routing matrix in Eq. (11) is a) the bi-adjacency matrix of the bipartite graph presented in Figure 4 and b) this bipartite graph is a $(2, 2, 1/4)$ -expander graph. For instance, let $\Phi = \{l_1, l_2\}$. In that case, $N(\{l_1, l_2\})$,

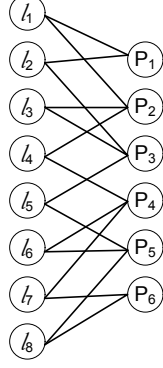


Fig. 4. Bipartite graph corresponding to routing matrix in Eq. (11)

the set of nodes connected to $\{l_1, l_2\}$ is:

$$N(\{l_1, l_2\}) = \{P_1, P_2, P_3\} \quad (12)$$

which means $|N(\{l_1, l_2\})| = 3$ and Eq. (7) holds with equality when $\epsilon = 1/4$.

The null space of matrix in Eq. (11) of dimension two, is spanned by the two following vectors:

$$\begin{aligned} \mathbf{w}_1 &= [0.36 \quad -0.36 \quad -0.23 \quad -0.14 \quad 0.59 \quad -0.23 \quad 0.36 \quad -0.36]^t \\ \mathbf{w}_2 &= [-0.18 \quad 0.18 \quad -0.45 \quad 0.63 \quad 0.26 \quad -0.45 \quad -0.18 \quad -0.18]^t \end{aligned} \quad (13)$$

Since $\epsilon = 1/4$, by Lemma 1, for any \mathbf{w} such that $\mathbf{R}\mathbf{w} = \mathbf{0}$, we have:

$$\|\mathbf{w}_S\|_1 \leq 0.5 \|\mathbf{w}_{S^c}\|_1 \quad (14)$$

To demonstrate this, let $S = \{1\}$, and $\mathbf{w} = \mathbf{w}_1 + 0.3\mathbf{w}_2$. Clearly \mathbf{w} is in null space of \mathbf{R} ; i.e. $\mathbf{w} \in \mathcal{N}(\mathbf{R})$. Then \mathbf{w} , \mathbf{w}_S and \mathbf{w}_{S^c} are given by:

$$\begin{aligned} \mathbf{w} &= [0.31 \quad -0.31 \quad -0.36 \quad 0.05 \quad 0.67 \quad -0.36 \quad 0.31 \quad -0.31]^t \\ \mathbf{w}_S &= [0.31 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^t \\ \mathbf{w}_{S^c} &= [0 \quad -0.31 \quad -0.36 \quad 0.05 \quad 0.67 \quad -0.36 \quad 0.31 \quad -0.31]^t \end{aligned} \quad (15)$$

Thus, $\|\mathbf{w}_S\|_1 = 0.31$ and $\|\mathbf{w}_{S^c}\|_1 = 2.36$ and inequality (14) is satisfied.

Let the true delay vector be:

$$\mathbf{x} = [0.1 \quad 0.1 \quad 0.1 \quad 0.1 \quad 1 \quad 0.1 \quad 0.1 \quad 0.1]^t \quad (16)$$

Then the measured vector using the measurement matrix in Eq. (11) is

$$\mathbf{y} = \mathbf{R}\mathbf{x} = [0.2 \quad 0.3 \quad 1.2 \quad 0.3 \quad 1.2 \quad 0.2]^t \quad (17)$$

Since $\mathbf{R}[\mathbf{w}_1 \ \mathbf{w}_2] = [\mathbf{0} \ \mathbf{0}]$, any vector of the form

$$\mathbf{x}' = \mathbf{x} + \alpha \mathbf{w}_1 + \beta \mathbf{w}_2 \quad (18)$$

results in the same \mathbf{y} . Note that α and β must be selected such that all entries of \mathbf{x}' are positive. For example, let $\alpha = \beta = 0.1$. Then

$$\mathbf{x}' = [0.12 \ 0.08 \ 0.03 \ 0.15 \ 1.08 \ 0.03 \ 0.12 \ 0.08]^t \quad (19)$$

Since $\epsilon = 1/4$, we have $f(\epsilon) = 1.5$. Let $S = \{5\}$ be the largest element in \mathbf{x} , then from Theorem 1 we have:

$$\begin{aligned} \|\mathbf{x}' - \mathbf{x}\|_1 = 0.34 &\leq f(\epsilon) \|\mathbf{x}_{S^c}\|_1 \\ &= 1.5 * (0.1 + 0.1 + 0.1 + 0.1 + 0.1 + 0.1 + 0.1) \\ &= 1.05 \end{aligned} \quad (20)$$

■

B. Network Identifiability

We use the result of previous sections to show that under some conditions, the link delay in a network may be estimated as the unique solution to Eq. (1). The following theorem relates the problem of delay estimation in a network $N(V, E)$ to results on expander graphs. In addition, it proves that Eq. (1) can be solved for \mathbf{x} using a linear program (LP).

Theorem 2. *Let $N(V, E)$ be a network with set of paths \mathcal{R} and corresponding routing matrix \mathbf{R} . Suppose $G(E, \mathcal{R}, H)$ is a bipartite graph with bi-adjacency matrix \mathbf{R} . Assume \mathbf{x} is the true delay vector of $N(V, E)$. Let \mathbf{x}^* be a solution to the following LP optimization:*

$$\min \|\mathbf{x}\|_1 \quad (21)$$

s.t.

$$\mathbf{R}\mathbf{x}^* = \mathbf{R}\mathbf{x}$$

Then

$$\|\mathbf{x} - \mathbf{x}^*\|_1 \leq f(\epsilon) \|\mathbf{x}_{S^c}\|_1 \quad (22)$$

if G is a $(2, d, \epsilon)$ -expander with $\epsilon \leq \frac{1}{4}$.

Proof: See Appendix A.

First note that if the true delay vector \mathbf{x} is 1-sparse, it implies $\|\mathbf{x}_{S^c}\|_1 = 0$ which means l_1 -norm minimization in Eq. (21) can recover \mathbf{x} with zero estimation error. However, if the true delay vector contains links with small

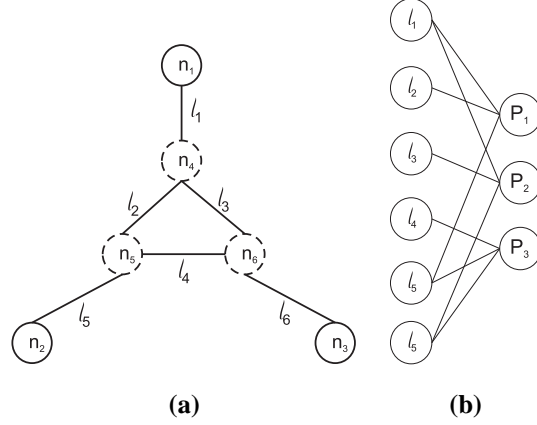


Fig. 5. A 1-identifiable network whose corresponding bipartite graph is not regular on the left side: (a) Network topology (b) Corresponding bipartite graph

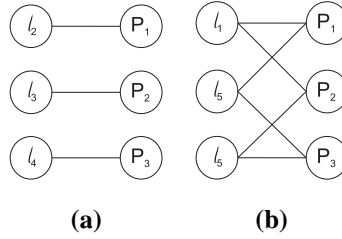


Fig. 6. Two subgraphs of the bipartite graph in Figure 5-b which are regular on their left side

but non-zero delays (the more likely scenario), the estimation error is not zero and the above theorem yields an upper bound.

It should be noted that in general, the reverse of Theorem 2 is not true; i.e. there exist some networks $N(V, E)$ which are 1-identifiable but their corresponding bipartite graph is not an expander graph. An example of such a network is depicted in Figure 5-(a) with the following routing matrix:

$$\mathbf{R} = \begin{matrix} & \begin{matrix} P_1 : n_1 \rightsquigarrow n_2 \\ P_2 : n_1 \rightsquigarrow n_3 \\ P_3 : n_2 \rightsquigarrow n_3 \end{matrix} & \begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{matrix} \quad (23)$$

The routing matrix above is a bi-adjacency matrix of the bipartite graph presented in Figure 5-(b). This bipartite graph is not regular in its left side since the degree of a node in left set is either one or two, and hence it cannot be an expander. However Figs. 6-(a) and (b), respectively, represent subgraphs of G with regular left degree one and two; each of these subgraphs are expander graphs. The above observation motivates us that the result in Theorem 2 may be extendable for networks whose corresponding bipartite graph is not regular (and therefore is not expander) but can be partitioned into sub-graphs that are expander graphs.

Theorem 3. Let $N(V, E)$ be a network with routing matrix \mathbf{R} . Let $G(X, Y, H)$ be a bipartite graph with bi-adjacency matrix \mathbf{R} . Suppose $G_i(X_i, Y, H_i)$, $i = 1, 2, \dots, M$ be d_i -regular bipartite subgraphs of G such that:

- $X = \cup X_i$
- $H = \cup H_i$
- $d_i \neq d_j$ for $i \neq j$

Then, $N(V, E)$ is 1-identifiable, if each of G_i is a $(2, d_i, \epsilon)$ -expander graph for $\epsilon \leq \frac{1}{4}$. Further, link delay is the solution to LP optimization in Eq. (21).

Proof: See Appendix A.

It should be noted that the condition $d_i \neq d_j$ implies that X_i 's corresponding to different G_i 's are disjoint. For future reference, we refer to the conditions in Theorem 3 as *1-identifiability expansion conditions* and if network $N(V, E)$ satisfy those conditions we refer to it as *1-identifiable expander network*.

Note that the 1-identifiable expansion conditions in Definition 3 implies the following for any link pair l_i and l_j :

- They belong to different G_i 's and hence have different degrees $d_i \neq d_j$;
- They belong to the same subgraph G_i ; i.e. $d_i = d_j$. In that case, since G_i is a bipartite graph, they satisfy the expansion property in Eq. (7).

We state this observation formally in the following corollary.

Corollary 4. Let $N(V, E)$ be a network with routing matrix \mathbf{R} and set of paths \mathcal{R} . Let $G(E, \mathcal{R}, H)$ be its corresponding bipartite graph with bi-adjacency matrix \mathbf{R} . Then one and only one of the following statement is true for any two links l_i and l_j in E , $i \neq j$:

- $\deg(l_i) > \deg(l_j)$
- $\deg(l_i) < \deg(l_j)$
- $\deg(l_i) + \deg(l_j) - 4\deg(l_i, l_j) \geq 0$

where $\deg(l_i, l_j)$ is defined as number of nodes connected to both l_i and l_j in the bipartite graph $G(E, \mathcal{R}, H)$ ¹.

proof: See Appendix A.

Theorem 2 (3) requires the graph G (G_i) to be a $(2, d, \epsilon)$ -expander. To the best of our knowledge, there is no existing algorithm in the literature to determine the expansion factor or error parameter of a given bipartite graph. In next section, we introduce a heuristic algorithm to determine if a given bipartite graph is a $(2, d, \epsilon)$ -expander and its error parameter ϵ .

Finally, for completeness, we state the following definition for k -identifiable expander network.

Definition 3. A k -identifiable expander $N(V, E)$ is a network whose routing matrix \mathbf{R} is the bi-adjacency matrix of a bipartite graph $G(X, Y, H)$ consisting of d_i -regular subgraphs $G(X_i, Y, H_i)$ with the following properties

¹It also can be stated as number of paths going through both links l_i and l_j in the network $N(V, E)$ with routing matrix \mathbf{R} .

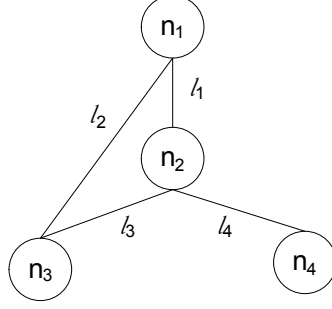


Fig. 7. There is only one walk of length 3 from n_1 to n_4 , $n_1l_2l_3l_4n_4$

- $X = \cup X_i$
- $H = \cup H_i$
- $d_i \neq d_j$ for $i \neq j$
- $G(X_i, Y, H_i)$ is a $(2k, d_i, \epsilon)$ -expander with $\epsilon \leq \frac{1}{4}$

C. How to determine an Expander Graph

In this section, we show how to determine if a given bipartite graph is a $(2, d, \epsilon)$ -expander graph and its error factor ϵ . Suppose $G(V, E)$ is a graph with adjacency matrix \mathbf{T} . A walk from node v to node u is a sequence of edges and nodes from v to u [14]. For example, a walk from node n_1 to n_4 is $n_1l_1n_2l_4n_4$ in Figure 7. The length of a walk is defined as the number of links in the walk. For example, in Figure 7 there are two walks with length one and two from n_1 to n_3 , $n_1l_2n_3$ and $n_1l_1n_2l_3n_3$, respectively.

The following theorem gives an algorithm to compute number of walks with length k between two nodes in $G(V, E)$ [28], [29].

Theorem 5. Let $G(V, E)$ be a graph with set of vertices $V = \{v_1, v_2, \dots, v_N\}$ and set of edges E , and let \mathbf{T} denote the adjacency matrix of G . Then the (i, j) th entry of \mathbf{T}^k is the number of different walks in G of length k from v_i to v_j .

For example, the graph in Figure 7 has the following adjacency matrix:

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (24)$$

Hence, \mathbf{T}^2 and \mathbf{T}^3 shows the number of walks between each pair of nodes with length two and three, respectively. Since

$$\mathbf{T}^3 = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 4 & 2 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 1 & 3 & 1 & 0 \end{bmatrix}, \quad (25)$$

there is only one walk with length three from n_1 to n_4 ($n_1 l_2 l_3 n_4$).

Now suppose $G(X, Y, H)$ is a bipartite graph with bi-adjacency matrix \mathbf{A} and adjacency matrix \mathbf{T} . The following relation between the bi-adjacency and adjacency matrix of a bipartite graph is given as follows:

$$\mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^t & \mathbf{0} \end{bmatrix} \quad (26)$$

Now consider two nodes $u, v \in X$. According to Theorem 5, the (u, v) -th entry in \mathbf{T}^2 shows the number of walks with length two from u to v . Since G is a bipartite graph, there is no connection between nodes in X . Thus, all paths from node u to node v must go through a node in Y that is connected to both u and v . Therefore, the number of walks with length two between u and v reveals the number of their common neighbors. This observation proves the following lemma:

lemma 2. *Let $G(X, Y, H)$ be a bipartite graph with biadjacency matrix \mathbf{A} . For any two nodes in X , the corresponding entry in $\mathbf{A}^t \mathbf{A}$ represents the number of neighbors in Y they have in common.*

For the bipartite graph in Figure 2, we have

$$\mathbf{A}^t \mathbf{A} = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix} \quad (27)$$

which means that any two nodes in the left hand side of the bipartite graph in Figure 2 have at most one node in common in the right hand side. Note that the diagonal entries of $\mathbf{A}^t \mathbf{A}$ represents the (common) degree of nodes in X . Since the graph is left 2-regular, it is easy to see that expansion is satisfied with $\epsilon = 1/4$. This leads to the following:

Theorem 6. *Let $G(X, Y, H)$ be a bipartite graph with bi-adjacency matrix \mathbf{A} and left regularity parameter d . Let λ be the maximum non-diagonal entry of matrix $\mathbf{A}^t \mathbf{A}$. Then G is a $(2, d, \epsilon)$ -expander where*

$$\epsilon = \frac{\lambda}{2d} \quad (28)$$

Up to this point, our goal was to determine if a network with a given routing matrix is identifiable or not. In practice, the additional probes required along these specified routes incur a cost in terms of overhead. In the next section, we investigate the problem of determining the *minimum number of paths* required for probing that are

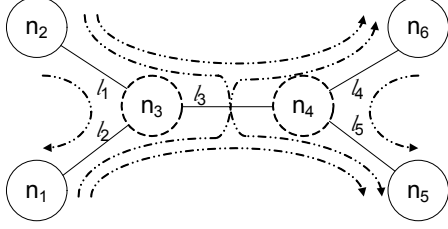


Fig. 8. Paths between boundary nodes of network in Figure 1

sufficient for network identifiability. Minimizing the number of paths used reduces number of probes injected into the network.

IV. MINIMUM PATH SET SELECTION

As an example, consider the network in Figure 1 which has 6 paths between boundary nodes as depicted in Figure 8; these are collected in the following set.

$$\mathcal{R} = \{l_1l_3l_4, l_1l_3l_5, l_2l_3l_4, l_2l_3l_5, l_1l_2, l_4l_5\} \quad (29)$$

However, only four of them are sufficient for delay identifiability of any single link. In general, not all paths between boundary nodes are necessary for identifiability. The next subsection shows how to select the set of paths from \mathcal{R} such that a desired objective is achieved.

A. Network Covering

Consider a network $N(V, E)$ with a collection of end-to-end paths \mathcal{R} . We seek the *minimum number of paths between boundary nodes such that each link belongs to at least one path*. This suffices to determine whether the network contains *any* bad link. This problem - known as 'network covering' - has been studied in literature [30], [5]. We provide a new algorithmic formulation in Section IV-B that finds the minimum number of paths needed for a given 1-identifiable network.

To formulate an optimization problem, define an indicator variable $I_{\mathcal{R}_i}$ indicating whether path $\mathcal{R}_i \in \mathcal{R}$ is included or not:

$$I_{\mathcal{R}_i} = \begin{cases} 1 & \mathcal{R}_i \text{ is used} \\ 0 & \text{o.w.} \end{cases} \quad (30)$$

We seek to minimize the number of paths; i.e.,

$$\min \sum_{i=1}^{|\mathcal{R}|} I_{\mathcal{R}_i}. \quad (31)$$

over all binary variables $I_{\mathcal{R}_i}$ subject to the fact that each link belongs to at least one of paths. Let $\mathbf{I}_{\mathcal{R}}^t = [I_{\mathcal{R}_i}]_1^{|\mathcal{R}|}$ be the vector of path indicators. The i -th entry of $\mathbf{R}^t \mathbf{I}_{\mathcal{R}}$ is the number of paths that traverse the i -th link, each of which should be equal to or greater than 1. We therefore seek to

$$\begin{aligned}
& \min \sum_{i=1}^{|\mathcal{R}|} I_{\mathcal{R}_i} \\
& \text{s.t.} \\
& \mathbf{R}^t \mathbf{I}_{\mathcal{R}} \geq 1 \\
& I_{\mathcal{R}_i} \in \{0, 1\}
\end{aligned} \tag{32}$$

The above is a *binary integer programming* which is well-known to be NP-hard; there exist a number of algorithms which provide good approximate solutions [31], [32] when the original problem can be ‘relaxed’.

B. Minimum Path Set for 1-identifiability

Among all available paths in the network, \mathcal{R} , what is the subset with minimum cardinality which guarantees 1-identifiability of the network? To achieve a network that is $k = 1$ identifiable, it is necessary that our minimum set ‘cover’ the network which means $\mathbf{R}^t \mathbf{I}_{\mathcal{R}} \geq 1$, as discussed in previous section. For identifiability, we use Corollary 4 that specifies conditions for any two links $l_i, l_j \in E$ to have the network 1-identifiable. Using the path indicator functions in Eq. (30), these conditions can be rewritten as below:

$$\begin{aligned}
& \bullet \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k} I_{\mathcal{R}_k} - \sum_{\mathcal{R}_k: l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} \geq 1 \\
& \bullet \sum_{\mathcal{R}_k: l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} - \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k} I_{\mathcal{R}_k} \geq 1 \\
& \bullet \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k} I_{\mathcal{R}_k} + \sum_{\mathcal{R}_k: l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} - 4 \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k, l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} \geq 0
\end{aligned} \tag{33}$$

As mentioned, for any two links l_i and l_j , one and only one of the above inequalities must be satisfied. Hence we use the notion of *alternative constraints* - a well-known device in linear binary programming [32]. Towards that end, we introduce three binary variables $y_{1ij}, y_{2ij}, y_{3ij}$ for any two links l_i and l_j , $i \neq j$:

$$y_{kij} = \begin{cases} 1 & \text{if the } k\text{-th constraint is satisfied} \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

Then we rewrite the constraints in Eq. (33) in the following format:

$$\begin{aligned}
& n(1 - y_{1ij}) + \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k} I_{\mathcal{R}_k} - \sum_{\mathcal{R}_k: l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} \geq 1 \\
& n(1 - y_{2ij}) + \sum_{\mathcal{R}_k: l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} - \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k} I_{\mathcal{R}_k} \geq 1 \\
& n(1 - y_{3ij}) + \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k} I_{\mathcal{R}_k} + \sum_{\mathcal{R}_k: l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} - 4 \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k, l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} \geq 0 \\
& y_{1ij} + y_{2ij} + y_{3ij} = 1
\end{aligned} \tag{35}$$

The last equality guarantees that exactly one of y_{kij} 's equals one and the others are zero. Note that if $y_{kij} = 0$, the k -th constraint is trivially satisfied. Therefore the constraints in Eq. (36) are equivalent to the statement that one and only one of in Eq. (33) holds. The above implies that if a subset of \mathcal{R} satisfies the constraints in Eq. (36), the network is 1-identifiable using those paths.

The following theorem summarized all of above findings for minimum number of paths which quarantines 1-identifiability of a given network $N(V, E)$.

Theorem 7. *Suppose a network $N(V, E)$ with routing matrix \mathbf{R} and equivalent set of paths \mathcal{R} is given. Determining the minimum number of paths r which guarantees 1-identifiability of the network can be cast as follows:*

$$\begin{aligned}
 & \min \sum_{i=1}^{|\mathcal{R}|} I_{\mathcal{R}_i} \\
 & \text{s.t.} \\
 & \mathbf{R}^t \mathbf{I}_{\mathcal{R}} \geq 0 \\
 & \forall l_i, l_j \in E (l_i \neq l_j) \\
 & n(1 - y_{1ij}) + \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k} I_{\mathcal{R}_k} - \sum_{\mathcal{R}_k: l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} \geq 1 \\
 & n(1 - y_{2ij}) + \sum_{\mathcal{R}_k: l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} - \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k} I_{\mathcal{R}_k} \geq 1 \\
 & n(1 - y_{3ij}) + \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k} I_{\mathcal{R}_k} + \sum_{\mathcal{R}_k: l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} - 4 \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k, l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} \geq 0 \\
 & y_{1ij} + y_{2ij} + y_{3ij} = 1 \\
 & y_{kij} \in \{0, 1\} \quad k = 1, 2, 3 \\
 & I_{\mathcal{R}_i} \in \{0, 1\} \quad \forall i
 \end{aligned} \tag{36}$$

where n is number of links inside the network ($n = |E|$).

The number of constraints for the binary optimization problem in Eq. (36) is of order $O(n^2)$. However, if Eq. (36) doesn't have a feasible solution, it is possible that $N(V, E)$ is still 1-identifiable.

C. Heuristic Algorithm

Since the optimization problem in Eq. (32) and (36) is NP-complete, we introduce a heuristic algorithm, based on relaxing the binary constraints in (36) leading to a linear programming problem which is solvable in polynomial time. This is achieved by substituting the constraints $I_{\mathcal{R}_i} \in \{0, 1\}$ and $y_{kij} \in \{0, 1\}$ with $I_{\mathcal{R}_i} \in [0, 1]$ and $y_{kij} \in [0, 1]$, respectively.

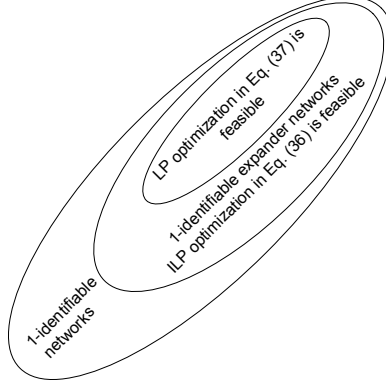


Fig. 9. Relation between feasibility in Eq. (21), in Eq. (36) and 1-identifiable networks

$$\begin{aligned}
 & \min \sum_{i=1}^{|\mathcal{R}|} I_{\mathcal{R}_i} \\
 & \text{s.t.} \\
 & \mathbf{R}^t \mathbf{I}_{\mathcal{R}} \geq 0 \\
 & \forall l_i, l_j \in E (l_i \neq l_j) \\
 & n(1 - y_{1ij}) + \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k} I_{\mathcal{R}_k} - \sum_{\mathcal{R}_k: l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} \geq 1 \\
 & n(1 - y_{2ij}) + \sum_{\mathcal{R}_k: l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} - \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k} I_{\mathcal{R}_k} \geq 1 \\
 & n(1 - y_{3ij}) + \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k} I_{\mathcal{R}_k} + \sum_{\mathcal{R}_k: l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} - 4 \sum_{\mathcal{R}_k: l_i \in \mathcal{R}_k, l_j \in \mathcal{R}_k} I_{\mathcal{R}_k} \geq 0 \\
 & y_{1ij} + y_{2ij} + y_{3ij} = 1 \\
 & y_{kij} \in [0, 1] \quad k = 1, 2, 3 \\
 & I_{\mathcal{R}_i} \in [0, 1] \quad \forall i
 \end{aligned} \tag{37}$$

Let $I_{\mathcal{R}}^*$ be the solution to above LP optimization and $I_{\mathcal{R}_{i^*}}^*$ the maximum entry in $I_{\mathcal{R}}^*$, i.e.,

$$I_{\mathcal{R}_{i^*}}^* = \arg \max_i I_{\mathcal{R}_i}^* \tag{38}$$

We use \mathcal{R}_{i^*} as one of the paths for probing or equivalently set $I_{\mathcal{R}_{i^*}}^* = 1$. Now we solve the LP in Eq. (37) with the additional constraint $I_{\mathcal{R}_{i^*}}^* = 1$. If we run the above algorithm $|\mathcal{R}|$ times, we achieve an approximate solution to the ILP problem in Eq. (36).

If the sequence of the LP problems are all feasible, it means there is a zero-one vector $\mathbf{I}_{\mathcal{R}}$ such that all the constraint in Eq. (36) are satisfied. This implies that the network $N(V, E)$ is a 1-identifiable expander and $\mathbf{I}_{\mathcal{R}}$ contains the requisite minimum number of paths. However, if one of the LP problem in (37) is not feasible, it

does not necessarily mean that the ILP is infeasible. In other words, the network $N(V, E)$ may be a 1-identifiable expander while sequence of LP problems in Eq. (37) is not feasible. Figure 9 summarizes the relation between the feasible sets for the LP optimization (37) and the ILP problem (21), and 1-identifiable networks.

V. EVALUATION RESULTS

In Section III we showed that if the routing matrix of a network is the bi-adjacency matrix for union of disjoint expander graphs, that network is 1-identifiable. Moreover, we can estimate internal link delay using a LP optimizer in Eq. (21). However, a legitimate ‘big-picture’ question arises: how many networks actually satisfy conditions of Theorem 3; i.e., are they 1-identifiable expanders? In this section, we generate random Internet-type networks to study this question. Our simulation results show that more than 70% of networks generated satisfy the conditions of Theorem 3 and therefore their link delay can be estimated using the l_1 minimization introduced in Eq. (21).

For those networks which are 1-identifiable expander, i.e., their routing matrix satisfies the condition in Definition 3 for $k = 1$, we determine average normalized estimation error when there is exactly $k = 1$ congested link ² in the network. Then we explore the average normalized estimation error when there are $k = 2, 3$ congested links inside the network - scenarios for which we do not have any analytical results to date. Surprisingly, even for $k > 1$, we find that the average normalized estimation error remains within an acceptable range. Finally, for those networks which are 1-identifiable expander, we determine the gain (relative to probing all possible paths in the network) by determining the histogram for minimum number of paths required to estimate link delay for a given error tolerance.

A. Generate Networks with Random Topology

We use Inet version 3.0 [33], [34] - an Internet topology generator software (at AS³ level) - to generate random graphs with given power law and fixed number of boundary nodes ⁴. We create networks containing 5000 nodes with 5, 8, 10, 12, 16 and 20 boundary nodes, respectively. The output of Inet, which gives neighbors of each node in the generated graph, is fed to matgraph toolbox in MATLAB [35] for modification. We first create a routing matrix containing the shortest paths between any boundary node pairs in the network. Then we delete all nodes and links which do not contribute to any of the above paths, since it is clear that if a link is not covered by any end-to-end path, it is not identifiable. Then we remove all nodes with degree two, as it is a well-known fact in network tomography that nodes with degree two are not 1-identifiable using end-to-end measurement techniques [36]. The remaining networks constitute our random set. In Figure 10, six examples of Internet-like random networks are depicted.

²A congested link is one with significantly elevated delay compared to the rest of the links in the network.

³Autonomous System

⁴Boundary nodes are nodes with degree one which act as injection point for probes in our problem.

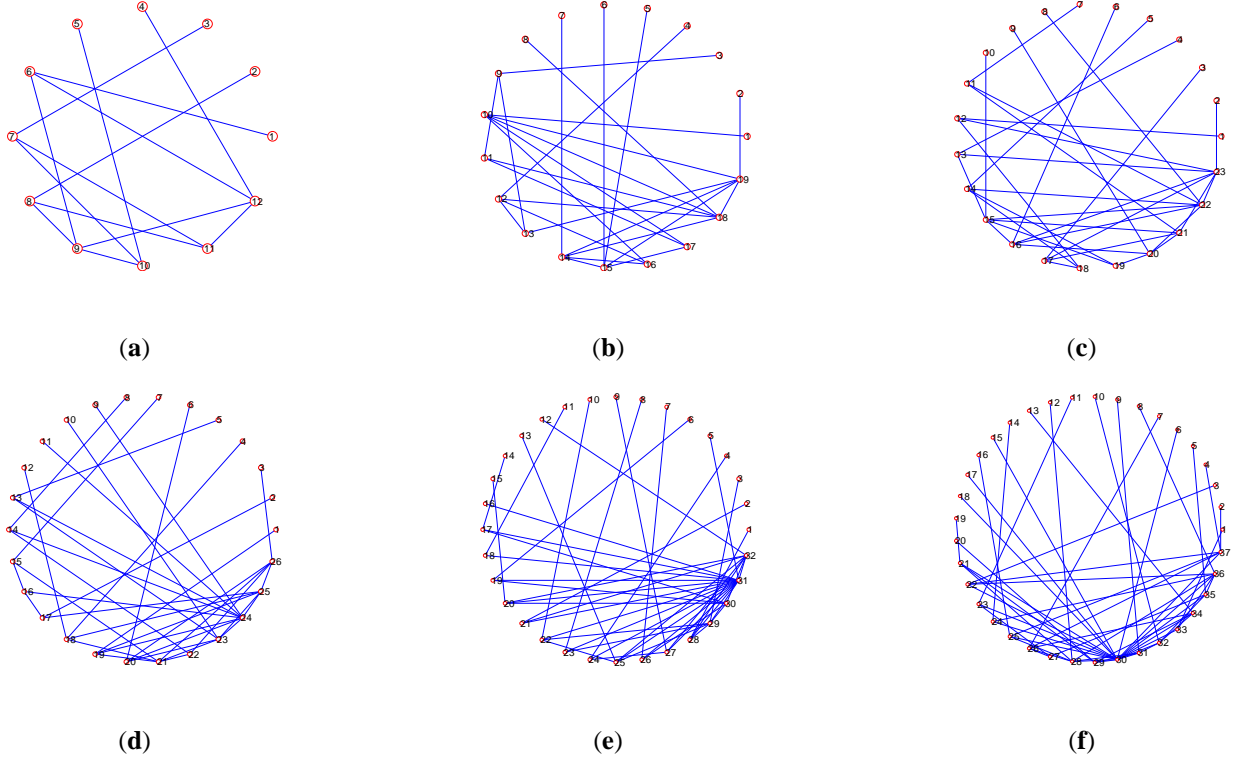


Fig. 10. Output of Inet after our modification in MATLAB with (a) 5 (b) 8 (c) 10 (d) 12 (e) 16 (f) 20 boundary nodes. Nodes with degree 1 represents injection nodes

B. Networks and Expansion Property

For the routing matrices of these random networks, we first examine how many of them satisfy the k -identifiability expansion conditions in Definition 3. For networks with fixed number of boundary nodes, fifty different topologies are created. Table II shows percentage of them which satisfy the k -identifiability expansion property for $k = 1, 2, 3$. Based on this observation, we can conclude that more than 70% of any Internet-topology based network satisfies expansion conditions given in Theorem 3 and therefore our results are applicable to them. As can be seen, the number of networks satisfying the expander conditions decreases as k increases. However, our simulation results in the next section shows the unexpected result that even if the delay vector \mathbf{x} is k -sparse with $k > 1$, the estimation error by using the proposed approach in Eq. (21) remains in an acceptable range.

To see how many networks satisfy 1-identifiable expander conditions, we use the algorithm in Table I. For a given network $N(V, E)$ with routing matrix $\mathbf{R}_{r \times n}$, this algorithm simply decomposes the routing matrix \mathbf{R} into sub-matrices $\mathbf{R}'_{r_i \times n_i}$ such that number of ones in each column of sub-matrix $\mathbf{R}'_{r_i \times n_i}$ are equal and different from any other sub-matrix $\mathbf{R}'_{r_j \times n_j}$, $i \neq j$. Then it checks if every sub-matrix $\mathbf{R}'_{r_i \times n_i}$ satisfies condition in Theorem 6 for $\epsilon \leq 1/4$.

For $k > 1$, as mentioned before, there is no algorithm in the literature. Hence, we use exhaustive search to check if network $N(V, E)$ satisfies k -identifiable expander conditions using the Definitions 3 and 2, which has

TABLE I

Algorithm : 1-identifiable networks:

Given network $N(V, E)$ with routing matrix $\mathbf{R}_{r \times n}$, \mathbf{r}_k is the k -th row of \mathbf{R} .

If $N(V, E)$ is 1-identifiable expander eflag is 1, otherwise it is zero.

```

eflag  $\leftarrow$  1
degs  $\leftarrow \sum_{k=1}^r \mathbf{r}_k$ 
for  $i$  in degs
  index  $\leftarrow$  find(degs ==  $i$ )
   $\mathbf{R}' \leftarrow \mathbf{R}(:, \text{index})$ 
   $\mathbf{A} \leftarrow \mathbf{R}'^t \cdot \mathbf{R}'$ 
  for  $j=1$  to dim( $\mathbf{A}$ )
     $\mathbf{A}(j,j) \leftarrow 0$ 
  end
   $\lambda \leftarrow \max(\mathbf{A})$ 
   $\epsilon \leftarrow \lambda/(2i)$ 
  if  $\epsilon > 1/4$ 
    eflag  $\leftarrow$  0
  end
end
end

```

TABLE II

FOR NETWORKS WITH FIXED NUMBER OF BOUNDARY NODES WHAT PERCENTAGE ARE K-IDENTIFIABLE EXPANDERS FOR $k = 1, 2, 3$

	5	8	10	12	16	20
$k = 1$	80%	82%	76%	72%	74%	72%
$k = 2$	0%	60%	62%	56%	50%	50%
$k = 3$	0%	0%	0%	56%	50%	46%

complexity of $O(|E|^{2k})$ that increases exponentially with k . Fortunately, it is not necessary to check if a network is k -identifiable expander for $k > 1$. Even if the delay vector \mathbf{x} is $k > 1$ sparse, the LP optimization (Eq. (21)) based solution in Theorem 3 can still be used. Our results in Table II show that the resulting normalized estimation error as shown in Fig. 11 remains acceptable.

As we discussed in Section III-B, the reverse of Theorem 3 is not true, i.e., there are 1-identifiable networks which do not satisfy expansion property. However, our simulation results suggests that such networks are a small fraction of all networks with Internet topology. In other words, if a network is 1-identifiable, with high probability (more than .7) it is 1-identifiable expander; i.e. satisfies conditions in Theorem 3.

C. Delay Estimation: Simulation Experiments

Theorem 3 says that if routing matrix of a network satisfies 1-identifiable expander conditions, then link delays in the network can be estimated using Eq. (21). To examine the accuracy of the proposed delay estimation method, for each network created in Section V-A, we calculated the average normalized estimation error for all links as

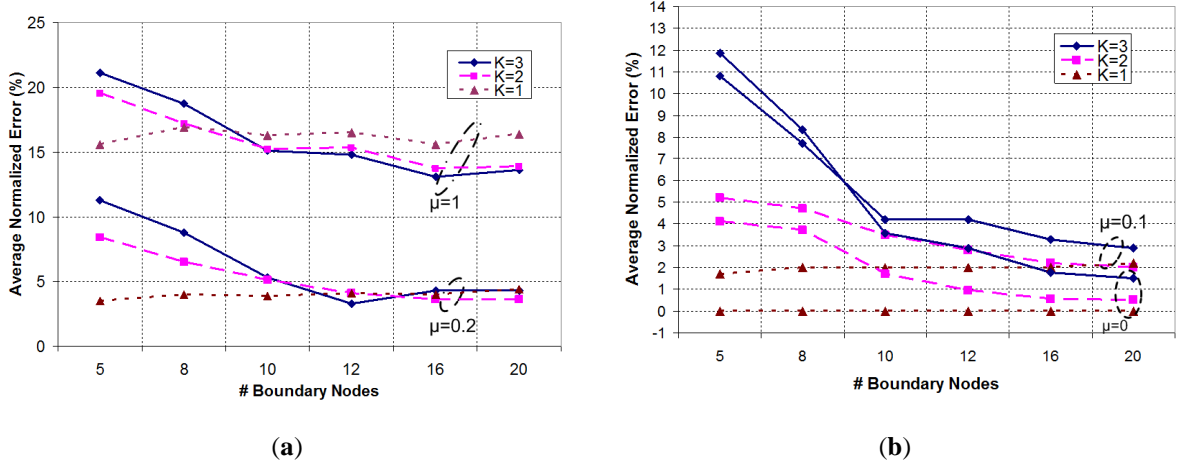


Fig. 11. Average normalized estimation error in networks satisfying conditions given in Theorem 3 when there are k deficient link within the network for different average delay μ in Eq. (39).

follows. A reference link is selected and assigned a delay of 10 ms. to denote that it is congested. All other links in the network are assumed to experience i.i.d exponentially distributed delays with average μ , i.e.,

$$f_l(t) = \frac{1}{\mu} \exp\left(-\frac{t}{\mu}\right) \quad \forall l \in E \quad (39)$$

where $f_l(t)$ is the delay for link l and $\mu \in [0, 1]$ to denote that these links do not undergo congestion.

We exploit the proposed LP optimization in (21) to estimate link delays. For the network, the normalized estimation error for each congested link inside the network is calculated as follows:

$$\text{norm. err} = \frac{\|\mathbf{x} - \hat{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2} \quad (40)$$

Figure 11 presents average normalized estimation error when there are k congested links inside the network for $k = 2, 3$ and LP optimization Eq. (21) is used to estimate the delay. As expected, for $\mu = 0$ (implying that the vector \mathbf{x} is 1-sparse), perfect recovery with zero error is possible, as shown by the corresponding plot in Fig. 11 (b). Otherwise, the average normalized estimation error for different μ mimics the expected trend from Eq. (22).

Although the result in Theorem 3 is derived based on $k = 1$, i.e. the vector delay \mathbf{x} in Eq. (1) is 1-sparse, the average normalized estimation error remains in an acceptable range for $k = 2$ and $k = 3$ deficient links. In fact for $k > 1$, the average normalized estimation error has a decreasing trend and for large networks, it is almost the same as $k = 1$ case. As Table II shows, the probability of being a k -identifiable expander is higher for a large network than a smaller one for $k = 2, 3$, supporting the decrease with k . In addition, for a large network with $k > 1$ deficient links, the deficient links are far from each other with high probability. Hence the network can be broken into k smaller sub-networks networks, each with one deficient link each, and is the possible reason why for large networks, the average normalized estimation error for $k > 1$ is almost the same as $k = 1$ case.

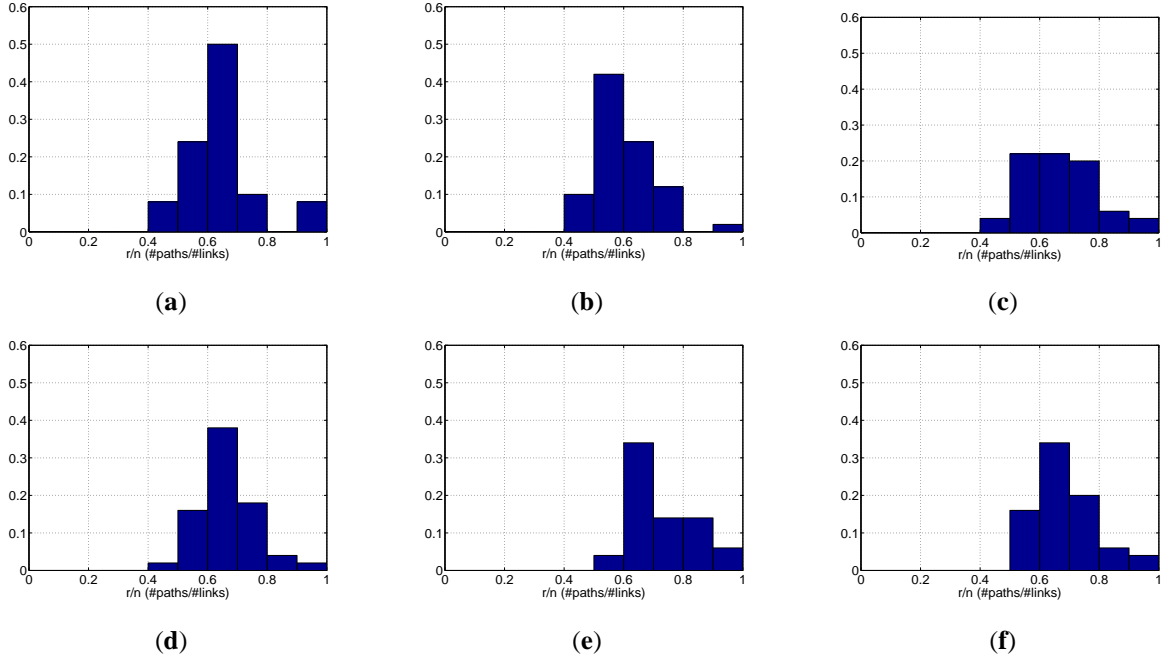


Fig. 12. Histogram of $\frac{r}{n}$ for 1-identifiable networks with (a) 5 (b) 8 (c) 10 (d) 12 (e) 16 (f) 20 boundary nodes.

D. Minimum Path Set

As mentioned in Section III, the goal of compressed sensing is to achieve maximum possible gain. In Section IV, based on our result on expander graphs, we proposed an optimization problem in Eq. (37) which gives the minimum number of paths (minimum r) that makes the network 1-identifiable. For networks with fixed number of boundary nodes which are 1-identifiable, Figure 12 shows a histogram of number of paths needed (relative to number of links in the network) in order to make the network 1-identifiable. After selecting paths which make the network 1-identifiable, the LP optimization proposed in Eq. (21) can be used to find the link delays.

As can be seen, in all the histograms in Figure 12, most of the mass is concentrated around 0.6. That means, if the network has n links, with high probability $r = 0.6n$ paths suffice to estimate link delay. Thus, the minimum path selection algorithm proposed in Section IV-B will save, with high probability, 40% percent of overhead probe traffic needed for link delay estimation compared with one where a probe is sent between all boundary pairs.

A second aspect highlighted by Figure 12 is that there is almost no mass below 0.4. This means that for a network satisfying conditions in Theorem 3, i.e. being a 1-identifiable expander, $\frac{r}{n}$ cannot be less than 0.4. Restated, to estimate links delay using proposed algorithm in Eq. (21) in a network with n links, the number of end-to-end measurements cannot be less than $0.4n$. The underlying cause is that the routing matrix of a graph exhibits structure distinct from a random binary network.

VI. CONCLUSION

This work presented a novel approach to estimate link delay in a network. Using the idea of binary compressed sensing, we developed an upper bound on the accuracy of delay recovery using an end-to-end probing method. Sending probes between nodes on the boundary of a network comes with the cost of increasing traffic inside the network. Thus, we showed how to design the routing matrix for a given network that minimizes the number of injected probes while preserving 1-identifiability. Future research directions include extending the analytical framework to networks which are $k > 1$ identifiable.

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APPENDIX A

PROOFS OF THEOREMS

Proof of Lemma 1:

Let \mathbf{A}' be submatrix of \mathbf{A} containing rows from $N(S)$. Since $|S| = 1$ and graph is left d -regular, $\|\mathbf{A}'\mathbf{w}_S\|_1 = \|\mathbf{A}\mathbf{w}_S\|_1 = d \|\mathbf{w}_S\|_1$. We have

$$\begin{aligned}
 0 = \|\mathbf{A}'\mathbf{w}\|_1 &= \|\mathbf{A}'\mathbf{w}_S + \mathbf{A}'\mathbf{w}_{S^c}\|_1 \\
 &\geq \|\mathbf{A}'\mathbf{w}_S\|_1 - \|\mathbf{A}'\mathbf{w}_{S^c}\|_1 \\
 &= d \|\mathbf{w}_S\|_1 - \|\mathbf{A}'\mathbf{w}_{S^c}\|_1.
 \end{aligned} \tag{41}$$

Each set of two nodes in the left part has at least $2(1-\epsilon)d$ neighbor nodes at the right side (expansion definition). Since each node at the left has degree d , number of common nodes is at most $2\epsilon d$. That means each column of \mathbf{A}' (except the one corresponding to S) has at most $2\epsilon d$ number of ones, yielding,

$$\|\mathbf{A}'\mathbf{w}_{S^c}\|_1 \leq 2\epsilon d \|\mathbf{w}_{S^c}\|_1 \tag{42}$$

Therefore

$$0 \geq d \|\mathbf{w}_S\|_1 - 2\epsilon d \|\mathbf{w}_{S^c}\|_1 \tag{43}$$

which means

$$\|\mathbf{w}_S\|_1 \leq 2\epsilon \|\mathbf{w}_{S^c}\|_1. \tag{44}$$

The above argument is valid if G is an expander graph. We now show that it implies $\epsilon \leq 1/4$. By definition of expander graph we have:

$$N(S) \geq 2(1 - \epsilon)d \quad (45)$$

To satisfy the expansion property, $N(S)$ needs to be strictly greater than d ; i.e. $N(S) \geq d + 1$. That means $2(1 - \epsilon)d$ must be at least $d + 1$. Hence, we have the following upper bound on ϵ

$$\epsilon \leq \frac{d - 1}{2d} \quad (46)$$

For an upper-bound that is independent of d , we find the infimum of right hand side and choose ϵ to satisfy that case. Clearly we have

$$\inf_{d=2,3,4,\dots} \frac{d - 1}{2d} = \frac{1}{4} \quad (47)$$

implying that $\epsilon \leq \frac{1}{4}$.

■

Theorem 1:

Let vector \mathbf{w} be in null space of \mathbf{A} ; i.e. $\mathbf{w} \in \mathcal{N}(\mathbf{A})$. Using Eq. (8) in lemma 1 we have:

$$\begin{aligned} \|\mathbf{w}_S\|_1 &\leq 2\epsilon \|\mathbf{w}_{S^c}\|_1 \\ \|\mathbf{w}_S\|_1 + 2\epsilon \|\mathbf{w}_S\|_1 &\leq 2\epsilon \|\mathbf{w}_{S^c}\|_1 + 2\epsilon \|\mathbf{w}_S\|_1 \\ \|\mathbf{w}_S\|_1 &\leq \frac{2\epsilon}{1 + 2\epsilon} \|\mathbf{w}\|_1 \end{aligned} \quad (48)$$

Now, let $\mathbf{y} = \mathbf{x}' - \mathbf{x}$. Clearly $\mathbf{y} \in \mathcal{N}(\mathbf{A})$ and we have:

$$\begin{aligned} \|\mathbf{x}\|_1 \geq \|\mathbf{x}'\|_1 &= \|(\mathbf{x} + \mathbf{y})_S\|_1 + \|(\mathbf{x} + \mathbf{y})_{S^c}\|_1 \\ &= \|\mathbf{x}_S + \mathbf{y}_S\|_1 + \|\mathbf{x}_{S^c} + \mathbf{y}_{S^c}\|_1 \\ &\geq \|\mathbf{x}_S\|_1 - \|\mathbf{y}_S\|_1 + \|\mathbf{y}_{S^c}\|_1 - \|\mathbf{x}_{S^c}\|_1 \\ &= \|\mathbf{x}\|_1 - 2\|\mathbf{x}_{S^c}\|_1 + \|\mathbf{y}\|_1 - 2\|\mathbf{y}_S\|_1 \\ &\geq \|\mathbf{x}\|_1 - 2\|\mathbf{x}_{S^c}\|_1 + (1 - \frac{4\epsilon}{1 + 2\epsilon}) \|\mathbf{y}\|_1 \end{aligned} \quad (49)$$

Where in the last equality we use result in Eq. (48). Therefore we have:

$$\|\mathbf{x}' - \mathbf{x}\|_1 = \|\mathbf{y}\|_1 \leq f(\epsilon) \|\mathbf{x}_{S^c}\|_1 \quad (50)$$

where $f(\epsilon) = \frac{2(1+2\epsilon)}{1-2\epsilon}$

■

⁵note that we exclude case of $d = 1$, since each bipartite graph which is left 1-regular is an expander graph

Theorem 2:

Let \mathbf{x}' be the solution to optimization problem in Eq. (21). It means $\mathbf{R}\mathbf{x}' = \mathbf{R}\mathbf{x}$ and $\|\mathbf{x}'\|_1 \leq \|\mathbf{x}\|_1$. On the other hand, G is a $(2, d, \epsilon)$ -expander graph with biadjacency matrix (\mathbf{R}) . Consequently Eq. (9) in Theorem 1 holds for \mathbf{x} and \mathbf{x}' .

■

Theorem 3:

We prove the theorem for the case in which $G(X, Y, H)$ has only two expander subgraphs. The general case can be easily extended following the same way. Let $G_1(X_1, Y, H_1)$ with $|X_1| = m$, and $G_2(X_2, Y, H_2)$ with $|X_2| = n - m$, be two d_i -regular ($d_1 \neq d_2$) subgraphs of $G(X, Y, H)$ with bi-adjacency matrices \mathbf{R}_1 and \mathbf{R}_2 , respectively. Without loss of generality, let rename the elements in X such that $\mathbf{R} = [\mathbf{R}_1 \mathbf{R}_2]$.

Now suppose $\mathbf{w} \in \mathbb{R}^n$ belong to null space of \mathbf{R} ; $\mathbf{w} = [\mathbf{w}_1^t \mathbf{w}_2^t]^t$.

Let S be any set of $k = 1$ coordinates of \mathbf{w} . Further, let \mathbf{R}' be submatrix of \mathbf{R} containing rows from $N(S)$. We consider two following cases:

Case 1: $S \subset \{1, 2, \dots, m\}$:

In this case S represents a node in $G_1(X_1, Y, H_1)$ which is a $(2, d_1, \epsilon)$ -expander by assumption. Similar to proof of Lemma 1, $\|\mathbf{R}'\mathbf{w}_S\|_1 = \|\mathbf{R}\mathbf{w}_S\|_1 = d_1 \|\mathbf{w}_S\|_1$. We have

$$\begin{aligned} 0 = \|\mathbf{R}'\mathbf{w}\|_1 &= \|\mathbf{R}'\mathbf{w}_S + \mathbf{R}'\mathbf{w}_{S^c}\|_1 \\ &\geq \|\mathbf{R}'\mathbf{w}_S\|_1 - \|\mathbf{R}'\mathbf{w}_{S^c}\|_1 \\ &= d_1 \|\mathbf{w}_S\|_1 - \|\mathbf{R}'\mathbf{w}_{S^c}\|_1 \end{aligned} \tag{51}$$

Since G_1 is left d_1 -regular \mathbf{R}' has d_1 rows. Using that, we can put an upper-bound on $\|\mathbf{R}'\mathbf{w}_{S^c}\|_1$ as follows. Let \mathbf{r}_i^t be the i -th rows of \mathbf{R}' . Then

$$\begin{aligned}
\| \mathbf{R}' \mathbf{w}_{S^c} \|_1 &= \sum_{i=1}^{d_1} |\mathbf{r}_i^t \mathbf{w}_{S^c}| \\
&= \sum_{i=1}^{d_1} \left| \sum_{j=1}^{|X|} r_{ij} w_{S^c j} \right| \\
&\stackrel{(1)}{\leq} \sum_{j=1}^{|X|} \sum_{i=1}^{d_1} r_{ij} |w_{S^c j}| \\
&= \sum_{j=1}^{|X|} \sum_{i=1}^{d_1} r_{ij} |w_{S^c j}| \\
&= \sum_{j=1}^{|X|} |w_{S^c j}| \sum_{i=1}^{d_1} r_{ij} \\
&= \sum_{j=1}^{|X_1|} |w_{S^c j}| \sum_{i=1}^{d_1} r_{ij} + \sum_{j=|X_1|+1}^{|X|} |w_{S^c j}| \sum_{i=1}^{d_1} r_{ij},
\end{aligned} \tag{52}$$

where for inequality (1), we used the triangular inequality and the fact that $r_{ij} \in \{0, 1\}$. Since $G_1(X_1, Y, H_1)$ is an $(2, d_1, \epsilon)$ -expander, each two nodes at the right hand side have at most $2\epsilon d_1$ neighbors at the right in common. That means, each column in \mathbf{R}' has at most $2\epsilon d$; i.e. $\sum_{i=1}^{d_1} r_{ij} \leq 2\epsilon d_1$ for each $i = \{1, 2, \dots, |X_1|\}$. On the other hand since \mathbf{R}' has d_1 rows, $\sum_{i=1}^{d_1} r_{ij} \leq d_1$ for each $i = \{|X_1| + 1, 2, \dots, |X|\}$. Using these facts and result in (52) we get:

$$\begin{aligned}
\| \mathbf{R}' \mathbf{w}_{S^c} \|_1 &\leq \sum_{j=1}^{|X_1|} |w_{S^c j}| \sum_{i=1}^{d_1} r_{ij} + \sum_{j=|X_1|+1}^{|X|} |w_{S^c j}| \sum_{i=1}^{d_1} r_{ij} \\
&\leq 2\epsilon d_1 \sum_{j=1}^{|X_1|} |w_{S^c j}| + d_1 \sum_{j=|X_1|+1}^{|X|} |w_{S^c j}| \\
&= 2\epsilon \| \mathbf{w}_{1S^c} \|_1 + d_1 \| \mathbf{w}_2 \|_1
\end{aligned} \tag{53}$$

Substituting above inequality in (51) we have:

$$0 \geq d_1 \| \mathbf{w}_S \|_1 - 2\epsilon d_1 \| \mathbf{w}_{1S^c} \|_1 - d_1 \| \mathbf{w}_2 \|_1 \tag{54}$$

Therefore we have the following upper-bound

$$\| \mathbf{w}_S \|_1 \leq 2\epsilon \| \mathbf{w}_{1S} \|_1 + \| \mathbf{w}_2 \|_1 \tag{55}$$

which also can be written as below:

$$\| \mathbf{w}_S \|_1 \leq \frac{2\epsilon}{1+2\epsilon} \| \mathbf{w}_1 \|_1 + \frac{1}{1+2\epsilon} \| \mathbf{w}_2 \|_1 \tag{56}$$

Case 2: $S \subset \{m+1, 2, \dots, n\}$:

By the same argument as Case 1, we have:

$$\begin{aligned}
0 = \| \mathbf{R} \mathbf{w} \|_1 &= \| \mathbf{R}' \mathbf{w} \|_1 \\
&\geq d_2 \| \mathbf{w}_S \|_1 - \| \mathbf{R}' \mathbf{w}_{S^c} \|_1
\end{aligned} \tag{57}$$

As in case 1, we can put an upper-bound on $\|\mathbf{R}'\mathbf{w}_{S^c}\|_1$ as follows.

$$\begin{aligned}
\|\mathbf{R}'\mathbf{w}_{S^c}\|_1 &\leq \sum_{j=1}^{|X_1|} |w_{S^c j}| \sum_{i=1}^{d_2} r_{ij} + \sum_{j=|X_1|+1}^{|X|} |w_{S^c j}| \sum_{i=1}^{d_2} r_{ij} \\
&\leq d_1 \sum_{j=1}^{|X_1|} |w_{S^c j}| + 2\epsilon d_2 \sum_{j=|X_1|+1}^{|X|} |w_{S^c j}| \\
&\leq d_1 \|\mathbf{w}_1\|_1 + 2\epsilon d_2 \|\mathbf{w}_{2S^c}\|_1
\end{aligned} \tag{58}$$

Using (57) and (58) we have the following upper-bound for $\|\mathbf{w}_S\|_1$.

$$\|\mathbf{w}_S\|_1 \leq 2\epsilon \|\mathbf{w}_{2S^c}\|_1 + \|\mathbf{w}_1\|_1 \tag{59}$$

which also can be written as below:

$$\|\mathbf{w}_S\|_1 \leq \frac{2\epsilon}{1+2\epsilon} \|\mathbf{w}_2\|_1 + \frac{1}{1+2\epsilon} \|\mathbf{w}_1\|_1 \tag{60}$$

Now suppose for a given \mathbf{y} , two 1-sparse vectors \mathbf{u} and \mathbf{v} satisfies equality $\mathbf{y} = \mathbf{R}\mathbf{u} = \mathbf{R}\mathbf{v}$. Let $\mathbf{w} = \mathbf{u} - \mathbf{v}$ and $\mathbf{u} = [\mathbf{u}_1^t \mathbf{u}_2^t]^t$, $\mathbf{v} = [\mathbf{v}_1^t \mathbf{v}_2^t]^t$ and $\mathbf{w} = [\mathbf{w}_1^t \mathbf{w}_2^t]^t$. Clearly, the following equalities hold:

$$\mathbf{w}_1 = \mathbf{u}_1 - \mathbf{v}_1 \tag{61}$$

$$\mathbf{w}_2 = \mathbf{u}_2 - \mathbf{v}_2$$

Without loss of generality, let's assume $\|\mathbf{u}\|_1 \geq \|\mathbf{v}\|_1$. We consider two following cases:

Case 1: $S \subset \{1, 2, \dots, m\}$

$$\begin{aligned}
\|\mathbf{u}\|_1 \geq \|\mathbf{v}\|_1 &= \|\mathbf{u}_1 + \mathbf{w}_1\|_1 + \|\mathbf{u}_2 + \mathbf{w}_2\|_1 \\
&= \|\mathbf{u}_{1S} + \mathbf{w}_{1S}\|_1 + \|\mathbf{u}_{1S^c} + \mathbf{w}_{1S^c}\|_1 + \|\mathbf{u}_2 + \mathbf{w}_2\|_1 \\
&\geq \|\mathbf{u}_{1S}\|_1 - \|\mathbf{w}_{1S}\|_1 + \|\mathbf{w}_{1S^c}\|_1 - \|\mathbf{u}_{1S^c}\|_1 + \|\mathbf{w}_2\|_1 - \|\mathbf{u}_2\|_1 \\
&= \|\mathbf{u}_{1S}\|_1 - (\|\mathbf{u}_{1S^c}\|_1 + \|\mathbf{u}_2\|_1) + (\|\mathbf{w}_{1S^c}\|_1 + \|\mathbf{w}_2\|_1) - \|\mathbf{w}_{1S}\|_1
\end{aligned} \tag{62}$$

Since $S \subset \{1, 2, \dots, m\}$, we have $\|\mathbf{w}_2\|_1 + \|\mathbf{w}_{1S^c}\|_1 = \|\mathbf{w}_{S^c}\|_1$ and $\|\mathbf{u}_2\|_1 + \|\mathbf{u}_{1S^c}\|_1 = \|\mathbf{u}_{S^c}\|_1$. So Eq. (62) can be simplified as bellow:

$$\begin{aligned}
2\|\mathbf{u}_{S^c}\|_1 &\geq \|\mathbf{w}_{S^c}\|_1 - \|\mathbf{w}_{1S}\|_1 \\
&= \|\mathbf{w}\|_1 - 2\|\mathbf{w}_{1S}\|_1
\end{aligned} \tag{63}$$

by using Eq. (56) we have:

$$2\|\mathbf{u}_{S^c}\|_1 \geq \frac{1-2\epsilon}{1+2\epsilon} \|\mathbf{w}_1\|_1 - \frac{1-2\epsilon}{1+2\epsilon} \|\mathbf{w}_2\|_1 \tag{64}$$

By Eq. (61) and triangular inequality we have:

$$\|\mathbf{w}_2\|_1 \leq \|\mathbf{u}_2\|_1 + \|\mathbf{v}_2\|_1 \tag{65}$$

Applying above inequality to (64) we have:

$$\frac{1+2\epsilon}{1-2\epsilon} \left[2 \| \mathbf{u}_{1S^c} \|_1 + \frac{1-2\epsilon}{1+2\epsilon} \| \mathbf{u}_2 \|_1 + \frac{1-2\epsilon}{1+2\epsilon} \| \mathbf{v}_2 \|_1 \right] \geq \| \mathbf{w} \|_1 \quad (66)$$

Clearly $\| \mathbf{u}_{S^c} \|_1 \geq \| \mathbf{u}_{1S^c} \|_1$, $\| \mathbf{u}_{S^c} \|_1 \geq \| \mathbf{u}_2 \|_1$ and $\| \mathbf{v}_{S^c} \|_1 \geq \| \mathbf{v}_2 \|_1$. Therefore, the following inequalities hold:

$$\frac{3+2\epsilon}{1-2\epsilon} \| \mathbf{u}_{S^c} \|_1 + \| \mathbf{v}_{2S^c} \|_1 \geq \| \mathbf{w} \|_1 \quad (67)$$

Now let $j \in S^c$. There is a path p^* which goes through link j and not link in S (since it is a logical network). Let \mathbf{r}_{p^*} be the corresponding row for p^* in routing matrix \mathbf{R} . Since $\mathbf{R}\mathbf{u} = \mathbf{R}\mathbf{v}$, we have:

$$\mathbf{r}_{p^*} \mathbf{u} = \mathbf{r}_{p^*} \mathbf{v} \geq v_j. \quad (68)$$

Since p^* doesn't go through link S , its corresponding entry in \mathbf{r}_{p^*} is zero. Hence we have $\| \mathbf{u}_{S^c} \|_1 \geq \mathbf{r}_{p^*} \mathbf{u}$. Therefore we can have the following upper-bound for every entry of $v_j \forall j \in S^c$

$$\| \mathbf{u}_{S^c} \|_1 \geq v_j \quad \forall j \in S^c \quad (69)$$

By adding up both side of the inequality for all $j \in S^c$ we have:

$$|S^c| \| \mathbf{u}_{S^c} \|_1 \geq \| \mathbf{v}_{S^c} \|_1. \quad (70)$$

Clearly $|X| > |S^c|$. Therefore, the following upper-bound is valid for $\mathbf{w} = \mathbf{u} - \mathbf{v}$:

$$\left(\frac{3+2\epsilon}{1-2\epsilon} + |X| \right) \| \mathbf{u}_{S^c} \|_1 \geq \| \mathbf{w} \|_1. \quad (71)$$

Case 2: $S \subset \{m+1, 2, \dots, n\}$. By the same argument as case 1 we have:

$$\left(\frac{3+2\epsilon}{1-2\epsilon} + |X| \right) \| \mathbf{u}_{S^c} \|_1 \geq \| \mathbf{w} \|_1. \quad (72)$$

Note that set X in $G(X, Y, H)$ is the same as E in network $N(V, E)$. The rest of the proof for LP optimization is the same as Theorem 2.

■

Proof of Corollary 4:

$N(V, E)$ with routing matrix \mathbf{R} is a 1-identifiable expander. That means bipartite graph $G(X, Y, H)$ with biadjacency matrix \mathbf{R} is union of left d_i -regular bipartite graphs $G(X_i, Y, H_i)$ such that:

- $X = \cup X_i$
- $H = \cup H_i$
- $d_i \neq d_j$ for $i \neq j$
- $G(X_i, Y, H_i)$ is a $(2, d_i, \epsilon)$ -expander with $\epsilon \leq \frac{1}{4}$

Now let consider two links l_i and l_j . If $\deg(l_i) \neq \deg(l_j)$ clearly one of the first two options in corollary 4 would be true. If $\deg(l_i) = \deg(l_j) = d_i$ they belong to the same subgraph, say, $G(X_i, Y, H_i)$. By the last condition of

1-identifiable network conditions, $G(X_i, Y, H_i)$ is a $(2, d, \epsilon)$ -expander. Now let $\Phi = l_i, l_j$. By definition of expander graph in Definition 2 The following holds for Φ :

$$|N(\Phi)| \geq (1 - \epsilon)d|\Phi| = 2(1 - \epsilon)d \quad (73)$$

By Theorem 2 maximum value possible for ϵ is $\frac{1}{4}$. Therefore minimum value for the right hand side of above inequality would be achieved if $\epsilon = \frac{1}{4}$ and it would be:

$$|N(\Phi)| \geq \frac{3}{2}d \quad (74)$$

$d(l_i, l_j)$ is defined to be number of nodes connected to both l_i and l_j . Using that we can calculate total number of nodes connected to at least one of l_i and l_j as follows:

$$|N(\Phi)| = d(l_i) + d(l_j) - d(l_i, l_j) = 2d - d(l_i, l_j) \quad (75)$$

Substituting above equality in Eq. (74) results in

$$d - 2d(l_i, l_j) \geq 0 \quad (76)$$

which can also be written as follows:

$$2d - 2d(l_i, l_j) = d(l_i) + d(l_j) - 4d(l_i, l_j) \geq 0 \quad (77)$$

■